Alpha as Ambiguity

Robust Mean-Variance Portfolio Analysis

F. Maccheroni, M. Marinacci, D. Ruffino
U. Bocconi, U. Bocconi, U. Minnesota

LSE

23 May 2012

The celebrated Arrow-Pratt approximation

$$u\ ^{1}\ E_Q\ \ u\ \ w\ \quad h \qquad w \quad E_Q\ \ h \quad \frac{\lambda_u\ \ w}{2}\sigma_Q^2\ \ h$$

has three main merits:

1. Theoretical identi…cation between risk and variance (risk management)
2. Theoretical identi…cation of risk aversion and the proportionality coe¢ cient $\lambda_u\ \ w$ (comparative statics)
3. Practical foundation for the preference model of investments' …nance

$$U\ \ X_{,} Q \qquad E_Q\ \ X \quad \frac{\lambda}{2}\sigma_Q^2\ \ X$$

(mean-variance utility)

Model Uncertainty a.k.a. Ambiguity

The amount of money $w \quad h$ is state contingent and for each model Q

$$c \;\; w \quad h, Q \qquad u^{1} \;\; E_Q \;\; u \;\; w \quad h \tag{1}$$

where u represents the agent's attitude toward state uncertainty .

If Q is unknown, then $c \;\; w \quad h,$ becomes a model contingent amount of money itself.

Suppose π to be the agent's prior probability on the possible models and v to be his attitude toward model uncertainty . earhoun(rd)]TJ/e3367k8samou

Setup

- $L^2\ P \quad L^2\ W, F, P$ square integrable random variables w.r.t. a reference model P (e.g., the physical measure)
- $I \quad R$ interval and $w \in \operatorname{int} I$
- $u, v : I \to R$ twice continuously di¤erentiable with $u', v' > 0$
- π Borel probability measure with bounded support on the models

$$D^2\ P \qquad Q \quad P : \frac{dQ}{dP} \in L^2\ P$$

with barycenter P, i.e., such that

$$\int_{D^2\ P} Q\ A\ d\pi\ Q \quad P\ A \quad \forall A \in F$$

Ambiguous expectations

For all $X \in L^2[P]$

$$E[X] : D^2[P] \to \mathbb{R}, \quad Q \mapsto E_Q[X]$$

is a continuous π-a.s. bounded function, with (second order) expectation

$$\int_{D^2[P]} E_Q[X]\, d\pi[Q] \qquad E_P[X]$$

and variance

$$\sigma^2_\pi[E[X]] \qquad \int_{D^2[P]} \left[E_Q[X] \qquad E_P[X] \right]^2 d\pi[Q]$$

Arrow-Pratt extended I

Theorem

For all P-a.s. bounded $h \; 2 \; L^2 \;\; P^{\;\;n}$ and $x \; 2 \; R^n$,

$$
\begin{aligned}
C \;\; w \quad x \;\; h \quad & w \quad E_P \;\; x \;\; h \quad \frac{\lambda_u \;\; w}{2} \sigma_P^2 \;\; x \;\; h && \text{(Arrow-Pratt)} \\
& \frac{\lambda_v \;\; w \qquad \lambda_u \;\; w}{2} \sigma_\pi^2 \;\; E \;\; x \;\; h && \text{(Ambiguity)} \\
& o \;\; jxj^2 && \text{(Remainder)}
\end{aligned}
$$

as $x \;\; ! \;\; 0$.

Arrow-Pratt extended II

For $n = 1$

$$C(w+h) \quad w + E_P[h] \quad \frac{l_u(w)}{2} s_P^2[h] \quad \frac{l_v(w) \quad l_u(w)}{2} s_p^2[E[h]]$$

As well known,risk aversion corresponds to $l_u(w) > 0$

Ceteris paribus, the greater $l_v(w)$

Unambiguous prospects

De…nition

X 2 $L^2(P)$ is (…rst moment) unambiguous

Classic risk theory

Theorem

Robust Mean-Variance preferences

An agent ranks prospects X in L^2 P by the following criterion

$$V \quad X \qquad E_P \quad X \qquad \frac{\lambda}{2}\sigma_P^2 \quad X \qquad \frac{\theta}{2}\sigma_\pi^2$$

The portfolio problem

- A unit of wealth has to be allocated among $n + 1$ assets at time 0
- The return on asset $i = 1, ..., n$, at time 1, is denoted by $r_i \in L^2(P)$. The $(n \times 1)$ vector of the returns is r and the $(n \times 1)$ vector of portfolio weights is w
- The return on the $(n + 1)$-th asset is risk-free, i.e. equal to a constant r_f
- The end-of-period return r_w, induced by a choice w, is

$$r_w = r_f + w' (r - r_f)$$

- Markets are frictionless

The optimal portfolio

The vector of portfolio weights w can be optimally chosen in R^n by solving

$$\max_{w \in R^n} V \quad r_w \qquad \max_{w \in R^n} \quad E_P \quad r_w \qquad \frac{\lambda}{2}\sigma_P^2 \quad r_w \qquad \frac{\theta}{2}\sigma_\pi^2 \quad E \quad r_w$$

Straightforward computation delivers the following optimality condition

$$\lambda Var_P \quad r \qquad \theta Var_\pi \quad E \quad r \quad \hat{w} \qquad E_P \quad r \qquad r_f \tag{3}$$

The most attractive feature of (3) is that it allows us to make use of the

One risky and one ambiguous assets

Portfolio weights

If $\sigma_P \;\; r_{e\,,}\, r_m \qquad 0$ then

$$\hat{w}_m \qquad \frac{E_P \;\; r_m \qquad r_f}{\lambda\sigma_P^2 \;\; r_m} \qquad \text{and} \qquad \hat{w}_e \qquad \frac{E_P \;\; r_e \qquad r_f}{\lambda\sigma_P^2 \;\; r_e \qquad \theta\sigma_\pi^2 \;\; E \;\; r_e}$$

Else if $\sigma_P \;\; r_{e\,,}\, r_m \;\; 6 \;\; 0$ then

$$\hat{w}_m \qquad \frac{E_P \;\; r_m \qquad r_f \qquad \lambda\sigma_P^2 \;\; r_e \qquad \theta\sigma_\pi^2 \;\; E \;\; r_e \qquad\qquad \lambda\sigma_P \;\; r_{e\,,}\, r_m \quad E_P \;\; r_e \qquad r_f}{\lambda^2\sigma_P^2 \;\; r_e \;\; \sigma_P^2 \;\; r_m \qquad \lambda\theta\sigma_\pi^2 \;\; E \;\; r_e \qquad {}^2}$$

Seeking the alpha

Our agent "seeks the alpha"

$$\operatorname{sgn} \hat{w}_e = \operatorname{sgn} a_{em}$$

Agent uses a_{em} as a criterion to decide whether to take a long or short position in the ambiguous asset, i.e., to decide in which side of the market of asset r_e

Reducing exposure

-